Chapter 8

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The world's population experiences exponential growth, which means that the rate of growth becomes more rapid as the size of the population increases. But how do we explain this in the language of calculus? Well, the rate of growth of the population is described by an exponential function, and the derivative of the population with respect to time is a constant multiple of the population. But there are other examples of growth that require not just exponential functions, but compositions of exponential functions with other functions. These examples include electronic signal transmission with amplification, the "bell curve" used in statistics, the effects of shock absorbers on car vibration, or the function describing population growth in an environment that has a maximum sustainable population. By combining the techniques in this chapter with other rules for derivatives, we can find the derivative of an exponential function that is composed with other functions. Logarithmic functions and exponential functions are inverses of each other, and in this chapter, you will see how their derivatives are also related to each other.

CHAPTER EXPECTATIONS In this chapter, you will

- identify e as $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^2$ and approximate the limit, **Section 8.1**
- define e and the derivative of $y = e^x$, **Section 8.1**
- define the logarithmic function $\log_a x$ $(a > 1)$, **Section 8.2**
- determine the derivatives of exponential and logarithmic function, **Section 8.2, 8.3, Career Link**
- determine the derivatives of combinations of the basic functions, **Section 8.3**
- solve optimization problems using exponential and logarithmic functions, **Section 8.4**
- make inferences from models of applications and compare the inferences with the original hypotheses regarding the rates of change, **Section 8.1, 8.2**
- compare the key features of a mathematical model with the features of the application it represents, **Section 8.3, 8.4**

Review of Prerequisite Skills

In Chapter 8, you will be studying two classes of functions that occur frequently in calculus problems: the derivatives of logarithmic and exponential functions. To begin, we will review some properties of exponential and logarithmic functions.

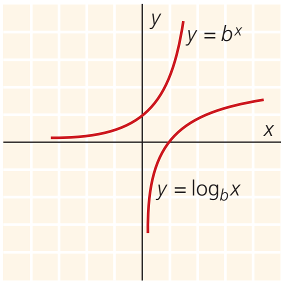
Properties of Logarithms

- $\log_b(pq) = \log_b p + \log_b q$
- $\log_b\left(\frac{p}{q}\right) = \log_b p - \log_b q$
- $\log_b(p^r) = r\log_b p$
- $\log_b(b^r) = r$

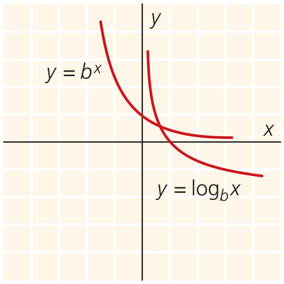
Properties of Exponents

- $b^m b^n = b^{m+n}$
- $\frac{b^m}{b^n} = b^{m-n}$
- $(b^m)^n = b^{mn}$
- $b^{\log_b(m)} = m$

The Graphs of $y = \log_b x$ and $y = b^x$



for $b > 1$



for $0 < b < 1$

- If $b^m = n$ for $b > 0$, then $\log_b n = m$.
- If $y = \log_a p$, then $y = \frac{\log_b p}{\log_b a}$ for any $a, b > 0$.

Exercise

1. Evaluate each of the following:

   a. 3^{-2} b. $32^{\frac{2}{5}}$ c. $27^{-\frac{2}{3}}$ d. $\left(\frac{2}{3}\right)^{-2}$

2. In each of the following, change to the equivalent logarithmic form.

   a. $5^4 = 625$ b. $4^{-2} = \frac{1}{16}$ c. $x^3 = 3$

   d. $10^w = 450$ e. $3^8 = z$ f. $a^b = T$

3. Express each of the following in an equivalent exponential form.

   a. $\log_{11}(121) = 2$ b. $\log_{125}(x) = \frac{1}{3}$ c. $\log_a 1296 = 4$ d. $\log_b A = W$

4. Evaluate each of the following:

   a. $\log_2 32$ b. $\log_{10} 0.0001$ c. $\log_{10} 20 + \log_{10} 5$

   d. $\log_2 20 - \log_2 5$ e. $3^{2\log_3 5}$ f. $\log_3\left(5^3 9^{-3} 25^{-\frac{3}{2}}\right)$

5. In each of the following, use the change of base formula to express the given logarithm in terms of the base b, and then use a calculator to evaluate to three decimal places.

   a. $\log_2(80)$, $b = e$ b. $3\log_5 22 - 2\log_5 15$, $b = 10$

6. Sketch the graph of each function and find its x-intercept.

   a. $y = \log_{10}(x + 2)$ b. $y = 5^{x+3}$

Career Link investigate

CHAPTER 8: RATE-OF-CHANGE MODELS IN MICROBIOLOGY

How would you find the slope of the function

$$y = \frac{(7x - 3)^{\frac{5}{2}}(3x + 2)^4}{}$$

using each of the Power, Product, Quotient, and Chain Rules? While this task would be very difficult using traditional methods of differentiation, it will be pain-free when you use the logarithmic and exponential differential calculus methods of this chapter. In addition to developing ideas and skills, you will also take the logarithmic and exponential models constructed in Chapters 6 and 7 and utilize them in rate-of-change applications.

Case Study — Microbiologist

Microbiologists contribute their expertise to many fields, including medicine, environmental science, and biotechnology. Enumerating, the process of counting bacteria, allows microbiologists to build mathematical models that predict populations. Once they can predict a population accurately, the model could be used in medicine, for example, to predict the dose of medication required to kill a certain bacterial infection. The data set in the table was used by a microbiologist to produce a polynomial-based mathematical model to predict population $p(t)$, as a function of time t, in hours, for the growth of a certain bacteria:

Time (in hours)	Population
0	1000
0.5	1649
1.0	2718
1.5	4482
2.0	7389

$$p(t) = 1000\left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5\right)$$

DISCUSSION QUESTIONS

1. How well does the equation fit the data set? Use the equation, a graph, and/or the graphing calculator to comment on the "goodness of fit."
2. What is the population after 0.5 h? How fast is the population growing at this time? (Use calculus to determine this.) Complete these calculations for the 1.0 h point.
3. What pattern did you notice in your calculations? Explain this pattern by examining the terms of this equation to find the reason why.

The polynomial function in this case is an approximation of the special function in mathematics, natural science, and economics, $f(x) = e^x$, where e has a value of 2.718 28…. At the end of this chapter, you will complete a task on rates of change of exponential growth in a biotechnology case study.

Section 8.1 — Derivatives of Exponential Functions

We are familiar with the properties of the exponential function $f(x) = b^x$, where $b > 0$. In trying to compute its derivative, we note that the Power Rule developed earlier does not apply, since the base of the exponential function is constant and the exponent varies. By using the definition of a derivative, we obtain

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{b^{x+h}-b^x}{h} \\
&= \lim_{h\to 0}\frac{b^x \bullet b^h - b^x}{h} && \text{(Properties of the exponential function)} \\
&= \lim_{h\to 0}\frac{b^x(b^h-1)}{h}. && \text{(Common factor)}
\end{aligned}
$$

The factor b^x is constant as $h \to 0$ and does not depend on h.

$$f'(x) = b^x \lim_{h\to 0}\frac{b^h-1}{h}$$

In fact,

$$
\begin{aligned}
f'(0) &= b^0 \lim_{h\to 0}\frac{b^h-1}{h} \\
&= \lim_{h\to 0}\frac{b^h-1}{h}.
\end{aligned}
$$

Therefore if $f(x) = b^x$,

$$f'(x) = b^x \bullet f'(0)$$

or

$$f'(x) = f(x)f'(0).$$

Here we have a surprising result. The derivative at any point is the product of the value of the function at that point and a constant. This constant is the value of the slope of the function at $x = 0$, namely $f'(0)$. In other words, the slope of the tangent line at a given point is proportional to the y-coordinate at that point.

It is clear that as b changes, the value of $f'(0)$ will change. Is there any value of b that gives a particularly useful result? The following investigation addresses this question.

INVESTIGATION

The purpose of this investigation is to examine the value of $f'(0) = \lim_{h\to 0}\frac{b^h-1}{h}$ for different values of b.

1. Using $h = 0.0001$, determine the value of $\lim_{h\to 0}\frac{b^h-1}{h}$ for $b = 1, 2, 3, 4, 5$, and 6.
2. What happens to the value of the expression as b increases?

3. a. What is the maximum value of b so that the $\lim_{h\to 0} \frac{b^h - 1}{h} < 1$?

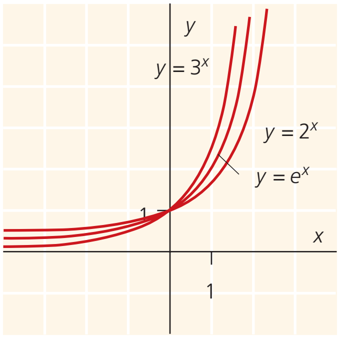
   b. What is the minimum value of b so that the $\lim_{h\to 0} \frac{b^h - 1}{h} > 1$?

4. What is the implication of $\lim_{h\to 0} \frac{b^h - 1}{h} = 1$ in calculating $f'(x) = f(x)f'(0)$?

5. Repeat Question 1 of this Investigation using $b = 2.5, 2.6, 2.7,$ and 2.8.

6. By further investigation, determine, correct to three decimal places, the value for b that gives a value of the limit closest to 1.

The exact number for which $f'(0) = 1$ is given the name "e," after the mathematician Euler (pronounced "oiler"). The approximate value of e is 2.718281. Its exact value cannot be determined, since e is an irrational number, as is π. With this number as base, we have the exponential function $f(x) = e^x$ with a particular property.



For the function $f(x) = e^x$, $f'(x) = e^x$.

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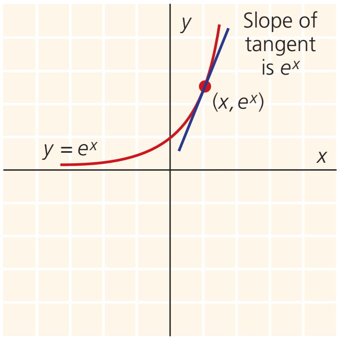
This function is its own derivative. The value of the slope of the tangent, at any point on the curve of $y = e^x$, is equal to the value of the y-coordinate at that point. There is a e^x LN button on your calculator. This button can be used to obtain the graph of $y = e^x$. Also note that entering e^1 yields 2.718281…, which is an approximation of the value of e. When we look at the graph of $y = e^x$, we see that the slopes of the tangents increase as x increases.



Technology Extension: Using a Spreadsheet Approach

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APPENDIX P. 447

To try this Investigation utilizing a spreadsheet option, please refer to page 447 of the Technical Assistance Appendix.

EXAMPLE 1 Find derivatives of the following functions:

a. $f(x) = x^2e^x$

b. $g(x) = e^{x^2 - x}$

Solution

a. Using the Product Rule,

$$\begin{aligned} f'(x) &= 2xe^x + x^2e^x \\ &= e^x(2x + x^2). \end{aligned}$$

b. If we let $u = x^2 - x$ and use the Chain Rule,

$$\begin{aligned} g(u) &= e^u \\ \frac{dg}{dx} &= \frac{dg}{du} \bullet \frac{du}{dx} \\ &= e^u(2x - 1) \\ &= e^{x^2 - x}(2x - 1). \end{aligned}$$

In general, if $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)}\, g'(x)$ by the Chain Rule.

EXAMPLE 2

Given $f(x) = 3e^{x^2}$, determine $f'(-1)$.

Solution

$$\begin{aligned} f'(x) &= 3e^{x^2}(2x) \\ &= 6xe^{x^2} \end{aligned}$$

Then $\qquad f'(-1) = -6e.$

Answers are usually left in this form. If desired, numeric approximations can be obtained from a calculator. Here $f'(-1) = -16.31$ correct to two decimals.

EXAMPLE 3

Determine the equation of the line tangent to the graph of $y = xe^x$ at the point where $x = 2$.

Solution

When $x = 2$, $y = 2e^2$, so $(2, 2e^2)$ is the point of contact of the tangent.

$$\begin{aligned} y' &= e^x + xe^x \\ &= e^x(1 + x) \end{aligned}$$

When $x = 2$, $y' = 3e^2$.

The equation of the tangent is $y - 2e^2 = 3e^2(x - 2)$

or $3e^2x - y - 4e^2 = 0$.

EXAMPLE 4

Determine the equation of the line tangent to the graph of $y = \dfrac{e^x}{x^2}$, $x \neq 0$, at the point where $x = 2$.

Solution

Using the Quotient Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{e^x x^2 - e^x(2x)}{(x^2)^2}\\ &= \frac{e^x x(x-2)}{x^4}\\ &= \frac{e^x(x-2)}{x^3},\ x \neq 0.\end{aligned}$$

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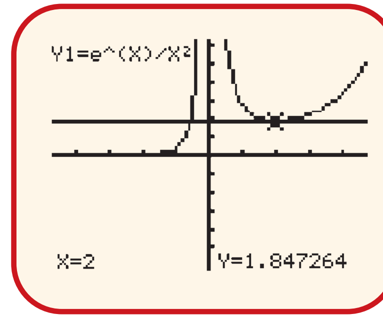
When $x = 2$, $\frac{dy}{dx} = 0$, and the tangent is horizontal. Therefore, the equation of the required tangent is $y = \frac{e^2}{4}$. A calculator yields this graph for $y = \frac{e^x}{x^2}$, and we see the horizontal tangent at $x = 2$.



We will return to the problem of finding the derivative of the general exponential function $f(x) = b^x$ in Section 8.3.

Exercise 8.1

Part A

1. If $f(x) = e^x$, compare the graphs of $y = f(x)$ and $y = f'(x)$.

Communication

2. Why can you not use the Power Rule for derivatives to differentiate $y = 2^x$?

3. Use the Chain Rule and the new Exponential Derivative Rule to find the derivative for each of the following:

a. $y = e^{3x}$	b. $s = e^{3t-5}$	c. $y = 2e^{10t}$
d. $y = e^{-3x}$	e. $y = e^{5-6x+x^2}$	f. $y = e^{\sqrt{x}}$

Knowledge/Understanding

4. Use the Exponential Derivative Rule in conjunction with other appropriate derivative rules to differentiate each of the following:

a. $y = 2e^{x^3}$	b. $y = xe^{3x}$	c. $f(x) = \frac{e^{-x^3}}{x}$	d. $s = \frac{e^{3t^2}}{t^2}$
e. $f(x) = \sqrt{x}e^x$	f. $h(t) = e^{t^2} + 3e^{-t}$	g. $p = e^{(w+e^w)}$	h. $g(t) = \frac{e^{2t}}{1+e^{2t}}$

5. a. If $f(x) = \frac{1}{3}(e^{3x} + e^{-3x})$, find $f'(1)$.
   b. If $f(x) = e^{-\left(\frac{1}{x+1}\right)}$, find $f'(0)$.
   c. If $h(z) = z^2(1 + e^{-z})$, determine $h'(-1)$.

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6. a. Find the equation of the tangent to the curve defined by $y = \frac{2e^x}{1 + e^x}$ at the point (0, 1).
   b. Use technology to graph the function in part **a** and draw the tangent at (0, 1).
   c. Compare the equation in part **a** to the computer equation.

Part B

Application

7. Find the equation of the tangent to the curve defined by $y = e^x$ that is perpendicular to the line defined by $3x + y = 1$.

8. Find the equation of the tangent to the curve defined by $y = xe^{-x}$ at the point $A(1, e^{-1})$.

9. Find all points at which the tangent to the curve defined by $y = x^2e^{-x}$ is horizontal.

10. If $y = \frac{5}{2}\left(e^{\frac{x}{5}} + e^{-\frac{x}{5}}\right)$, then prove that $y'' = \frac{y}{25}$.

11. a. For the function $y = e^{-3x}$, determine $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.
    b. From the pattern in part **a**, state the value for $\frac{d^ny}{dx^n}$.

12. For each of the following, determine the equation of the tangent at the given point.
    a. For the curve defined by $y - e^{xy} = 0$ at $A(0, 1)$.
    b. For the curve defined by $x^2e^y = 1$ at $B(1, 0)$.
    c. Explain why these relations cannot easily be graphed using a calculator.

Application

13. The number, N, of bacteria in a culture at time t in hours is
$N = 1000\left[30 + e^{-\frac{t}{30}}\right]$.
    a. What is the initial number of bacteria in the culture?
    b. Find the rate of change of the number of bacteria at time t.
    c. How fast is the number of bacteria changing when $t = 20$ h?
    d. Find the largest number of bacteria in the culture during the interval $0 \le t \le 50$.

14. The distance (in metres) fallen by a skydiver t seconds after jumping (and before her parachute opens) is $s = 160\left(\frac{1}{4}t - 1 + e^{-\frac{t}{4}}\right)$.

a. Find the skydiver's velocity, v, at time t.

b. Show that her acceleration is given by $a = 10 - \frac{1}{4}v$.

c. Find $v_T = \lim\limits_{t \to \infty} v$. This is the "terminal" velocity, the constant velocity attained when the air resistance balances the force of gravity.

d. At what time is the skydiver's velocity 95% of the terminal velocity? How far has she fallen at that time?

Part C

Thinking/Inquiry/ Problem Solving

15. Use the definition of the derivative to evaluate each limit.

a. $\lim\limits_{h \to 0} \frac{e^h - 1}{h}$

b. $\lim\limits_{h \to 0} \frac{e^{2+h} - e^2}{h}$

16. For what values of m does the function $y = Ae^{mt}$ satisfy the following equation?

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

17. The hyperbolic functions are defined as $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

a. Prove $D_x \sinh x = \cosh x$.

b. Prove $D_x \cosh x = \sinh x$.

c. Prove $D_x \tanh x = \frac{1}{(\cosh x)^2}$, if $\tanh x = \frac{\sinh x}{\cosh x}$.

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Graphing the Hyperbolic Function

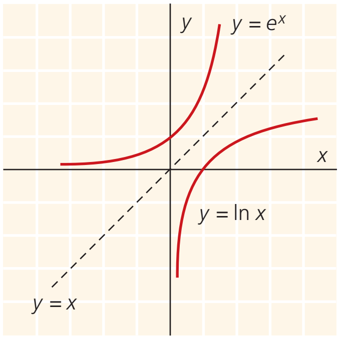
1. Use a calculator or computer to graph $y = \cosh x$ by using the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

2. Press 2nd CATALOG 0 for the list of CATALOG items and select **cosh(** to investigate if cosh is a built-in function.

3. On the same window, graph $y = 1.25x^2 + 1$ and $y = 1.05x^2 + 1$. Investigate changes in the coefficient a in the equation $y = ax^2 + 1$ to see if you can create a parabola that will approximate the hyperbolic cosine function.

Section 8.2 — The Derivative of the Natural Logarithmic Function

The logarithmic function is the inverse of the exponential function. For the particular exponential function $y = e^x$, the inverse is $x = e^y$ or $y = \log_e x$, a logarithmic function where $e \approx 2.719281$. This logarithmic function is referred to as the "natural" logarithmic function, and is usually written as $y = \ln x$ (pronounced "lon x"). The functions $y = e^x$ and $y = \ln x$ are inverses of each other. This means that the graphs of the functions are reflections of each other in the line $y = x$, as shown.



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There is a LN key on your calculator that is useful for sketching the graph of $y = \ln x$. This key can also be used to determine the numeric value of the natural logarithm of a number. For example, $\log 12 \approx 1.079$, while $\ln(12) \approx 2.485$.

What is the derivative of this logarithmic function? For $y = \ln x$, the definition of the derivative yields

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\ln(x + h) - \ln(x)}{h}.$$

We could investigate this limit in order to determine the value of $\frac{dy}{dx}$, an investigation we will consider later. First, we can determine the derivative of the natural logarithm function using the derivative of the exponential function we developed in the previous section.

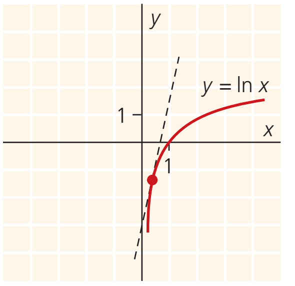
Given $y = \ln x$, we can rewrite this as $e^y = x$. Differentiating both sides of this equation with respect to x, and using implicit differentiation on the left side, yields

$$\begin{aligned} e^y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{e^y} \\ &= \frac{1}{x}. \end{aligned}$$

The derivative of the natural logarithmic function $y = \ln x$ is

$\frac{dy}{dx} = \frac{1}{x}, x > 0.$

This fits nicely with the graph of $y = \ln x$. The function is defined only for $x > 0$, and the slopes are all positive. We see that as $x \to \infty$, $\frac{dy}{dx} \to 0$. As x increases, the slope of the tangent decreases.



We can apply this new derivative, along with the Product, Quotient, and Chain Rules to find derivatives of fairly complicated functions.

EXAMPLE 1

Find $\frac{dy}{dx}$ for the following functions:

a. $y = \ln(5x)$ b. $y = \frac{\ln x}{x^3}$ c. $y = \ln(x^2 + e^x)$

Solution

a. $y = \ln(5x)$

Solution 1

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{1}{5x}(5) = \frac{1}{x}$$

Solution 2

$y = \ln(5x) = \ln(5) + \ln(x)$

$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

b. $y = \frac{\ln x}{x^3}$

Using the Quotient and Power Rules,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\ln(x)) \bullet x^3 - \ln(x)\frac{d}{dx}(x^3)}{(x^3)^2} \\ &= \frac{\frac{1}{x} \bullet x^3 - \ln(x) \bullet 3x^2}{x^6} \\ &= \frac{x^2 - 3x^2\ln(x)}{x^6} \\ &= \frac{1 - 3\ln(x)}{x^4}.\end{aligned}$$

c. $y = \ln(x^2 + e^x)$

Using the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{(x^2 + e^x)}\frac{d}{dx}(x^2 + e^x) \\ &= \frac{2x + e^x}{(x^2 + e^x)}.\end{aligned}$$

If $f(x) = \ln(g(x))$, then $f'(x) = \frac{1}{g(x)}g'(x)$, by the Chain Rule.

EXAMPLE 2

Determine the equation of the line tangent to $y = \frac{\ln x^2}{3x}$ at the point where $x = 1$.

Solution

$\ln 1 = 0$, so $y = 0$ when $x = 1$, and the point of contact of the tangent is $(1, 0)$.

The slope of the tangent is given by $\frac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{3x\left(\frac{1}{x^2}\right)2x - 3\ln x^2}{9x^2}\\ &= \frac{6 - 3\ln x^2}{9x^2}\end{aligned}$$

When $x = 1$, $\frac{dy}{dx} = \frac{2}{3}$.

The equation of the tangent is $y - 0 = \frac{2}{3}(x - 1)$, or $2x - 3y - 2 = 0$.

EXAMPLE 3

a. For the function $f(x) = \sqrt{x} - \ln x$, $x > 0$, use your graphing calculator to determine the value of x at a point on the graph that minimizes the function.

b. Use calculus methods to determine the exact solution.

Solution

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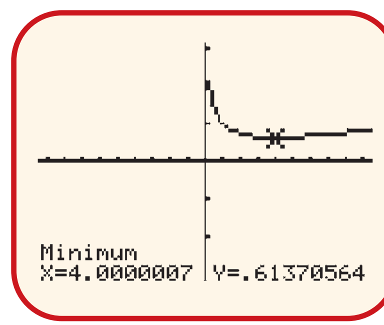
a. The graph of $f(x) = \sqrt{x} - \ln x$ is shown. Use the minimum value function, **3:minimum**, in the CALCULATE mode of your calculator to find the minimum value of $f(x)$. The minimum value occurs at $x = 4$.



b. $f(x) = \sqrt{x} - \ln x$

To minimize $f(x)$, set the derivative equal to zero.

$$\begin{aligned}f'(x) &= \frac{1}{2\sqrt{x}} - \frac{1}{x}\\ \frac{1}{2\sqrt{x}} - \frac{1}{x} &= 0\\ \frac{1}{2\sqrt{x}} &= \frac{1}{x}\\ x &= 2\sqrt{x}\\ x^2 &= 4x\\ x(x - 4) &= 0\\ x &= 4 \text{ or } x = 0\end{aligned}$$

But $x = 0$ is not in the domain of the function, so $x = 4$.
Therefore, the minimum value of $f(x)$ occurs at $x = 4$.

We now look back at the derivative of the natural logarithm function using the definition.

For the function $f(x) = \ln(x)$,

$$f'(x) = \lim_{h \to 0} \frac{\ln(x + h) - \ln(x)}{h}$$

and, specifically,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{\ln(1 + h) - \ln(1)}{h} \\ &= \lim_{h \to 0} \frac{\ln(1 + h)}{h} \\ &= \lim_{h \to 0} \ln(1 + h)^{\frac{1}{h}}, \text{ since } \frac{1}{h}\ln(1 + h) = \ln(1 + h)^{\frac{1}{h}}. \end{aligned}$$

However, since we know that $f'(x) = \frac{1}{x}$, $f'(1) = 1$.

Then $\lim_{h \to 0} \ln(1 + h)^{\frac{1}{h}} = 1$.

Since the natural logarithm function is a continuous and a one-to-one function, (meaning that for each acceptable value of the variable, there is exactly one function value), we can rewrite this as

$$\ln\left[\lim_{h \to 0} (1 + h)^{\frac{1}{h}}\right] = 1.$$

Since $\ln e = 1$,

$$\ln\left[\lim_{h \to 0} (1 + h)^{\frac{1}{h}}\right] = \ln e.$$

Therefore, $\lim_{h \to 0} (1 + h)^{\frac{1}{h}} = e$.

Earlier in this chapter, the value of e was presented as $e \approx 2.718281$. We now have a means of approximating the value of e using the above limit.

h	0.1	0.01	0.001	0.0001
$(1 + h)^{\frac{1}{h}}$	2.59374246	2.704813829	2.71692393	2.7181459268

From the table, it appears that $e \approx 2.718281$ is a good approximation as h approaches zero.

Exercise 8.2

Part A

Communication

1. Distinguish between natural logarithms and common logarithms.

2. At the end of this section, we found that we could approximate the value of e, Euler's constant, using $e = \lim_{h \to 0}(1 + h)^{\frac{1}{h}}$. By substituting $h = \frac{1}{n}$, we can express e as $e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$. Justify the definition by evaluating the limit for increasing values of n.

3. Use the Chain Rule in conjunction with the Logarithm Derivative Rule to find the derivative for each of the following:

a. $y = \ln(5x + 8)$ b. $y = \ln(x^2 + 1)$ c. $s = 5\ln t^3$

d. $y = \ln\sqrt{x + 1}$ e. $s = \ln(t^3 - 2t^2 + 5)$ f. $w = \ln\sqrt{z^2 + 3z}$

Knowledge/Understanding

4. Use the Logarithm Derivative Rule in conjunction with other appropriate derivative rules to differentiate each of the following:

a. $f(x) = x\ln x$ b. $y = \frac{\ln x}{x^2}$ c. $y = e^{\ln(x)}$

d. $y = [\ln x]^3$ e. $v = e^t \ln t$ f. $g(z) = \ln(e^{-z} + ze^{-z})$

g. $s = \frac{e^t}{\ln t}$ h. $h(u) = e^{\sqrt{u}} \ln \sqrt{u}$ i. $f(x) = \ln\left(\frac{x^2 + 1}{x - 1}\right)$

technology

5. a. If $g(x) = e^{2x-1}\ln(2x - 1)$, evaluate $g'(1)$.

b. If $f(t) = \ln\left(\frac{t - 1}{3t + 5}\right)$, evaluate $f'(5)$.

c. Check the above calculations using either a calculator or a computer.

6. For each of the following functions, solve the equation $f'(x) = 0$.

a. $f(x) = \ln(x^2 + 1)$

b. $f(x) = (\ln x + 2x)^{\frac{1}{3}}$

c. $f(x) = (x^2 + 1)^{-1}\ln(x^2 + 1)$

technology

7. a. Find the equation of the tangent to the curve defined by $f(x) = \frac{\ln\sqrt[3]{x}}{x}$ at the point where $x = 1$.

b. Use technology to graph the function in part **a**, and then draw the tangent at the point where $x = 1$.

c. Compare the equation in part **a** to the equation obtained on the calculator or computer.

Part B

Application **8.** Find the equation of the tangent to the curve defined by $y = \ln x - 1$ that is parallel to the straight line with equation $3x - 6y - 1 = 0$.

9. a. If $f(x) = (x \ln x)^2$, then find all points at which the graph of $f(x)$ has a horizontal tangent line.

technology

b. Use a calculator or a computer to check your work in part **a**.

c. Comment on the efficiency of the two solutions.

10. Find the equation of the tangent to the curve defined by $y = \ln(1 + e^{-x})$ at the point where $x = 0$.

Application **11.** The velocity in kilometres per hour of a car as it begins to slow down is given by the equation $v(t) = 90 - 30\ln(3t + 1)$, where t is in seconds.

a. What is the velocity of the car as the driver begins to brake?

b. What is the acceleration of the car?

c. What is the acceleration at $t = 2$?

d. How long does it take the car to stop?

12. The pH value of a chemical solution measures the acidity or alkalinity of the solution. The formula is

$$pH = -\log_{10}(H)$$
$$= -\frac{\ln(H)}{\ln(10)}$$ (Using the Change of Base identity)

where H is the concentration of hydrogen ions in the solution (in moles per litre).

a. Tomatoes have $H = 6.3 \times 10^{-5}$. Find the pH value.

b. Recipe ingredients are being added to a bowl of tomatoes, so that the concentration of hydrogen ions in the whole mixture is given by $H(t) = 30 - 5t - 25\left(e^{-\frac{t}{5}} - 1\right)$ moles per litre, where t is measured in seconds. Determine the rate of change of the pH value with respect to time after 10 s.

13. If a force F is defined by $F = k(e^{-S} - 6e^{-2S})$, where S is the distance between two objects, then prove $\frac{d^2F}{dS^2} = F - 18ke^{-2S}$.

14. For each of the following, find the slope of the tangent.

a. For the curve defined by $xe^y + y\ln x = 2$, at $(1, \ln 2)$.

b. For the curve defined by $\ln\sqrt{xy} = 0$, at the point $\left(\frac{1}{3}, 3\right)$.

Part C

Thinking/Inquiry/ Problem Solving

15. Use the definition of the derivative to evaluate $\lim_{h\to 0}\frac{\ln(2+h)-\ln(2)}{h}$.

16. One definition for e is $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

    a. Use the Binomial Theorem to expand $\left(1+\frac{1}{n}\right)^n$ and then evaluate the limit.

       Show that the value of e can be calculated from the series
       $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots.$

    b. For the above series calculate the following partial sums: S_3, S_4, S_5, S_6, and S_7.

17. Recall that $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

    a. Find the derivative of $y = \ln|x|$.

    b. Find the derivative of $y = \ln|2x+1|$.

    c. Find the derivative of $y = x^2\ln|x|$.

Section 8.3 — Derivatives of General Exponential and Logarithmic Functions

In finding the derivative of the general exponential function $f(x) = b^x$ in Section 8.1, we obtained the expression $f'(x) = b^x f'(0)$. We saw that there is a problem in determining $f'(0)$ for different values of the base b. We can avoid this problem by using the fact that $\frac{d}{dx}e^x = e^x$ together with the properties of logarithmic and exponential functions.

First, note that $e^{\ln(p)} = p$. Then for $f(x) = b^x$, we can write

$$\begin{aligned} f(x) &= b^x \\ &= e^{\ln(b^x)} \\ &= e^{x \ln b}. \quad \text{(Property of Logarithmic Functions)} \end{aligned}$$

Since $\ln b$ is a constant for any value of b, we differentiate $f(x) = e^{x \ln b}$ using the Chain Rule and get

$$\begin{aligned} f'(x) &= e^{x \ln(b)} \frac{d}{dx}(x \ln b) \\ &= e^{x \ln(b)} \ln b \\ &= e^{\ln(b^x)} \ln b \\ &= b^x \ln b. \end{aligned}$$

What happens if the exponent is $g(x)$? Then we have $f(x) = b^{g(x)}$. Now we let $u = g(x)$ and use the Chain Rule on $f(u) = b^u$, as follows:

$$\frac{df}{dx} = \frac{df}{du} \bullet \frac{du}{dx}.$$

Then $\frac{df}{du} = b^u \ln b$ and $\frac{du}{dx} = g'(x)$.

$$\frac{df}{dx} = b^{g(x)} \ln b[g'(x)]$$

For $f(x) = b^{g(x)}$, $f'(x) = b^{g(x)} \ln b(g'(x))$.

If $g(x) = x$ and $f(x) = b^x$, then $f'(x) = b^x \ln b$.

The derivative of $f(x) = 2^x$ is $f'(x) = 2^x \ln(2)$, and the derivative of $g(x) = 3^x$ is $g'(x) = 3^x \ln(3)$. (Check on your calculator to verify that $\ln 2 \approx 0.69$ and $\ln 3 \approx 1.09$, which match the values estimated in Section 8.1.)

Note that the derivative of $f(x) = e^x$ is a special case of this general derivative. If we apply the general formula, we get $f'(x) = e^x \ln(e) = e^x$, since $\ln e = 1$.

EXAMPLE 1 Find the derivative of the function $f(x) = 3^{x^2+2}$.

Solution

We have $f(x) = 3^{g(x)}$ with $g(x) = x^2 + 2$.
Then $g'(x) = 2x$.
Now, $f'(x) = 3^{x^2+2}$ (ln 3)$2x$.

We can use these derivatives to solve problems.

EXAMPLE 2 On January 1, 1850, the population of Goldrushtown was 50 000. Since then the size of the population has been modelled by the function $P(t) = 50\,000(0.98)^t$, where t is the number of years since January 1, 1850.

a. What was the population of Goldrushtown on January 1, 1900?
b. At what rate was the population of Goldrushtown changing on January 1, 1900? Was it increasing or decreasing at that time?

Solution

a. January 1, 1900 is exactly 50 years after January 1, 1850, so we let $t = 50$.

$$\begin{aligned} P(50) &= 50\,000(0.98)^{50} \\ &= 18\,208.484 \end{aligned}$$

The population on January 1, 1900 was approximately 18 208.

b. To determine the rate of change of the population, we require the derivative of P.

$$\begin{aligned} P'(t) &= 50\,000(0.98)^t \ln(0.98) \\ P'(50) &= 50\,000(0.98)^{50} \ln(0.98) \\ &\approx -367.861 \end{aligned}$$

Hence, after 50 years, the population was decreasing at a rate of approximately 368 people per year. (We expected the rate of change to be negative, as the original population function was a decaying exponential function with a base less than 1.)

We can now determine the derivative of the general logarithmic function, that is $y = \log_b x$, for any base $b > 0$. We use the fact that this logarithmic function can be written as $b^y = x$. Differentiating both sides implicitly with respect to x yields

$$\begin{aligned} b^y \ln b \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{b^y \ln b} \\ &= \frac{1}{x \ln b}. \end{aligned}$$

For the special case of the natural logarithm function, that is, $y = \log_e x = \ln x$, this becomes $\frac{dy}{dx} = \frac{1}{x \ln e} = \frac{1}{x}$.

In the general case, for $y = \log_b g(x)$, $b^y = g(x)$, from the definition of logarithms.

Taking derivatives $b^y \ln b \dfrac{dy}{dx} = g'(x)$,

$$\frac{dy}{dx} = \frac{g'(x)}{b^y \ln b} = \frac{g'(x)}{g(x)\ln b}.$$

For $y = \log_b x$, $\dfrac{dy}{dx} = \dfrac{1}{x\ln b}$.

For $y = \log_e x$, $\dfrac{dy}{dx} = \dfrac{1}{x}$.

For $y = \log_b g(x)$, $\dfrac{dy}{dx} = \dfrac{g'x}{g(x)\ln b}$.

EXAMPLE 3 Find the value of $g'(2)$ for the function $g(x) = x^3\log_2\left(\sqrt{x^2+4}\right)$.

Solution

Using the Product and Chain Rules,

$$\begin{aligned} g'(x) &= 3x^2\log_2\left(\sqrt{x^2+4}\right) + x^3 \frac{1}{\quad\quad} \frac{2x}{\quad\quad} \\ &= 3x^2\log_2\left(\sqrt{x^2+4}\right) + \frac{x^4}{(x^2+4)\ln 2}. \end{aligned}$$

Then

$$\begin{aligned} g'(2) &= 12\log_2\sqrt{4+4} + \frac{16}{(4+4)\ln(2)} \\ &= 12\log_2\sqrt{8} + \frac{16}{8\ln 2} \\ &= 12\log_2 2^{\frac{3}{2}} + \frac{2}{\ln 2} \\ &= 12 \times \frac{3}{2} + \frac{2}{\ln 2} = 18 + \frac{2}{\ln 2}. \end{aligned}$$

EXAMPLE 4 If I_0 is the intensity of barely audible sound, then the decibel (dB) measure of the loudness, L, of a sound of intensity I is given by $L = \log_{10}\left(\frac{I}{I_0}\right)$.

a. What is the loudness, in dB, of a sound of intensity I_0? $100\ I_0$?

b. If the intensity of a siren is 7000 units and is increasing at 100 000 units per second, what is the accompanying rate of change of the loudness of the sound?

Solution

a. A sound of intensity I_0 has loudness

$$\begin{aligned} L &= \log_{10}\left(\frac{I_0}{I_0}\right) \\ &= \log_{10} 1 \\ &= 0 \text{ dB}. \end{aligned}$$

A sound of intensity 100 I_0 has loudness

$$L = \log_{10}\left(\frac{100I_0}{I_0}\right)$$
$$= \log_{10}(100)$$
$$= 2 \text{ dB.}$$

b. We know that $\frac{dI}{dt} = 100\,000$ and we wish to find $\frac{dL}{dt}$.

$$L = \log_{10}\left(\frac{I}{I_0}\right)$$

Differentiate both sides of the equation implicitly with respect to time, t, and get

$$\frac{dL}{dt} = \frac{1}{\frac{I}{I_0}\ln 10}\left(\frac{1}{I_0}\frac{dI}{dt}\right) \qquad \textit{Note: } \frac{d}{dt}\left(\frac{I}{I_0}\right) = \frac{1}{I_0}\frac{dI}{dt}$$
$$= \frac{I_0}{I\ln 10}\left(\frac{1}{I_0}\frac{dI}{dt}\right)$$
$$= \frac{1}{I\ln(10)}\frac{dI}{dt}.$$

When $I = 7000$ and $\frac{dI}{dt} = 100\,000$, we get

$$\frac{dL}{dt} = \frac{1}{7000\ln(10)}(100\,000)$$
$$\approx 6.2.$$

Therefore, the loudness of the siren is increasing at approximately 6.2 dB per second.

In this section, we learned the derivatives for the general exponential and logarithmic functions $y = b^x$ and $y = \log_b(x)$. Many students will wish to compile a list of formulas and memorize them. However, there is no need to do so. You can always use identities to convert general exponential and logarithmic functions to exponential and logarithmic functions with base e, and then use the specific derivatives of $y = e^x$ and $y = \ln(x)$ along with the Chain Rule.

Exercise 8.3

Part A

Knowledge/Understanding

1. Use the Chain Rule in conjunction with the Exponential or Logarithm Derivative Rule to find the derivative for each of the following:

a. $y = 2^{3x}$

b. $y = 3.1^x + x^3$

c. $s = 10^{3t-5}$

d. $w = 10^{(5-6n+n^2)}$

e. $y = \log_5(x^3 - 2x^2 + 10)$

f. $y = \log_{10}\left(\frac{1+x}{1-x}\right)$

g. $f(x) = 7^{x^2}$

h. $v = \log_2\sqrt{t^2 + 3t}$

i. $y = 3^{x^2+3}$

2. Find the derivative function for each of the following:

a. $v = \frac{2^t}{t}$ b. $y = 2^x \log_2(x^4)$ c. $p = 2\log_3(5^s) - \log_3(4^s)$

d. $s = t^2 \log_{10}(1 - t)$ e. $f(x) = \frac{\sqrt{3^x}}{x^2}$ f. $y = \frac{\log_5(3x^2)}{\sqrt{x + 1}}$

3. a. If $f(t) = \log_2\left(\frac{t + 1}{2t + 7}\right)$, evaluate $f'(3)$.

b. If $h(t) = \log_3[\log_2(t)]$, determine $h'(8)$.

4. If $f(t) = 10^{3t-5} \bullet e^{2t^2}$, then find the values of t so that $f'(t) = 0$.

Communication

5. a. Find the equation of the tangent to the curve defined by $y = 10^{2x-9}\log_{10}(x^2 - 3x)$ at the point where $x = 5$.

b. Use technology to graph the function and the tangent defined in part **a**.

technology

c. Compare the equation determined by the calculator with the theoretical result. If you did not have the theoretical equation, explain how you would know when the equation provided by the calculator (or computer) is accurate to three decimal places.

Part B

6. Find the equation of the tangent to the curve defined by $y = 20 \times 10^{\left(\frac{t-5}{10}\right)}$ at the point where the curve crosses the y-axis.

Communication

7. Let $f(x) = \log_2(\log_2(x))$.

a. Find the domain of $f(x)$.

b. Find the slope of the tangent to the graph of $f(x)$ at its x-intercept.

c. Explain why it is extremely difficult to check this result on a calculator.

8. A particle moves according to the formula $s = 40 + 3t + 0.01t^2 + \ln t$, where t is measured in minutes and s in centimetres from an initial point I.

a. Find the velocity after 20 min.

b. Determine how long it takes before the acceleration of the particle is 0.01 cm/min^2.

9. Historical data shows that the amount of money sent out of Canada for interest and dividend payments during the period 1967 to 1979 can be approximated by the model $P = 0.5(10^9)e^{0.20015t}$, where t is measured in years, ($t = 0$ in 1967) and P is the total payment in Canadian dollars.

a. Determine and compare the rates of increase for the years 1968 and 1978.

b. Assume the model remained accurate until the year 2002. Compare the rate of increase for 1988 to the increase in 1998.

c. Investigate: Check the Web site of Statistics Canada to see if the rates of increase predicted by this model were accurate for 1988 and 1998.

Part C

Thinking/Inquiry/ Problem Solving

10. An earthquake of minimal intensity, I_0, is given a value 0 on the Richter scale. An earthquake of intensity I has a magnitude $R = \log_{10}\left(\frac{I}{I_0}\right)$ on the Richter scale.

a. Show that increasing the intensity of the earthquake by a factor of 10 will increase the Richter magnitude by 1.

b. If the intensity I of a large earthquake is increasing at a rate of 100 units per second, at what rate is the Richter magnitude increasing when $I = 35$?

11. In this section, we used an identity property to rewrite b^x as $e^{x\ln(b)}$. This allowed us to easily determine the derivative of b^x using the known derivative of e^x. This alternate expression for b^x is also useful when we are asked to sketch a graph of $y = b^x$. Since our calculator does not have exponential keys for a general base b, we rewrite $y = e^{x\ln(b)}$ and use the e^x key to sketch the graph.

a. On your calculator, sketch the graph of $y = e^x$.

b. On the same set of axes, use your calculator to sketch the graph of $y = 7^x$.

c. Give a graphical interpretation of the factor ln 7 in the expression $7^x = e^{x\ln(7)}$.

12. The change of base formula allows us to rewrite $\log_b x$ as $\frac{\ln x}{\ln b}$. Again, this is useful when graphing $y = \log_b(x)$ on a calculator, since calculators do not have logarithmic keys for a general base b.

a. On your calculator, sketch the graph of $y = \ln x$.

b. On the same set of axes, use your calculator to sketch the graph of $y = \log_5 x$.

c. Give a graphical interpretation of the factor $\frac{1}{\ln 5}$ in the expression $\log_5(x) = \frac{\ln x}{\ln 5}$.

Section 8.4 — Optimization Problems

In earlier chapters, you considered numerous situations in which you were asked to optimize a given situation. Recall that to optimize means to determine values of variables so that a function that represents cost, area, number of objects, or other quantities can be minimized or maximized.

Here we will consider further optimization problems, using exponential and logarithmic functions.

EXAMPLE 1

The effectiveness of studying for a test depends on how many hours a student studies. Some experiments showed that if the effectiveness, E, is put on a scale of 0 to 10, then $E(t) = 0.5\left[10 + te^{-\frac{t}{20}}\right]$, where t is the number of hours spent studying for an examination. If a student has up to 30 h that he can spend studying, how many hours should he study for maximum effectiveness?

Solution

We wish to find the maximum value for the function $E(t) = 0.5\left[10 + te^{-\frac{t}{20}}\right]$ on the interval $0 \le t \le 30$.

First find critical points by determining $E'(t)$.

$$\begin{aligned} E'(t) &= 0.5\left(e^{-\frac{t}{20}} + t\left(-\frac{1}{20}e^{-\frac{t}{20}}\right)\right) \qquad \text{(Using the Product and Chain Rules)} \\ &= 0.5e^{-\frac{t}{20}}\left(1 - \frac{t}{20}\right) \end{aligned}$$

E' is never undefined, and $e^{-\frac{t}{20}} > 0$ for all values of t. Then, $E'(t) = 0$ when

$$\begin{aligned} 1 - \frac{t}{20} &= 0 \\ t &= 20. \end{aligned}$$

To determine the maximum effectiveness, we use the algorithm for extreme values.

$$\begin{aligned} E(0) &= 0.5\left(10 + 0e^{0}\right) = 5 \\ E(20) &= 0.5\left(10 + 20e^{-1}\right) \doteq 8.7 \\ E(30) &= 0.5\left(10 + 30e^{-1.5}\right) \doteq 8.3 \end{aligned}$$

Therefore, the maximum effectiveness measure of 8.7 is achieved when a student studies 20 h for the exam.

EXAMPLE 2

A mathematical consultant determines that the proportion of people who will have responded to the advertisement of a new product after it has been marketed for t days is given by $f(t) = 0.7\left(1 - e^{-0.2t}\right)$. The area covered by the advertisement contains 10 million potential customers, and each response to the advertisement results in revenue to the company of \$0.70 (on average), excluding the cost of advertising. The advertising costs \$30 000 to produce and a further \$5000 per day to run.

a. Find $\lim\limits_{t \to \infty} f(t)$ and interpret the result.

b. What percentage of potential customers have responded after seven days of advertising?

c. Write the function $P(t)$ that represents the profit after t days of advertising. What is the profit after seven days?

d. For how many full days should the advertising campaign be run in order to maximize the profit? Assume an advertising budget of \$180 000.

Solution

a. As $t \to \infty$, $e^{-0.2t} \to 0$, so $\lim\limits_{t \to \infty} f(t) = \lim\limits_{t \to \infty} 0.7\left(1 - e^{-0.2t}\right) = 0.7$. This result means that if the advertising is left in place indefinitely (forever), 70% of the population will respond.

b. $f(7) = 0.7\left(1 - e^{-0.2(7)}\right) \approx 0.53$

After seven days of advertising, 53% of the population has responded.

c. The profit is the difference between the revenue received from all customers responding to the ad minus the advertising costs. Since the area covered by the ad contains 10 million potential customers, the number of customers responding to the ad after t days is

$10^7\left[0.7\left(1 - e^{-0.2t}\right)\right] = 7 \times 10^6\left(1 - e^{-0.2t}\right).$

The revenue to the company from these respondents is

$R(t) = 0.7\left[7 \times 10^6\left(1 - e^{-0.2t}\right)\right] = 4.9 \times 10^6\left(1 - e^{-0.2t}\right).$

The advertising costs for t days are $C(t) = 30\,000 + 5000t$.

Therefore, the profit to the company after t days of advertising is given by

$$\begin{aligned} P(t) &= R(t) - C(t) \\ &= 4.9 \times 10^6\left(1 - e^{-0.2t}\right) - 30\,000 - 5000t. \end{aligned}$$

After seven days of advertising, the profit is

$$\begin{aligned} P(7) &= 4.9 \times 10^6\left(1 - e^{-0.2(7)}\right) - 30\,000 - 5000(7) \\ &\approx 3\,627\,000. \end{aligned}$$

d. If the total advertising budget is \$180 000, then we require that

$$\begin{aligned} 30\,000 + 5000t &\leq 180\,000 \\ 5000t &\leq 150\,000 \\ t &\leq 30. \end{aligned}$$

We wish to maximize the profit function $P(t)$ on the interval $0 \leq t \leq 30$. For critical points, determine $P'(t)$.

$$P'(t) = 4.9 \times 10^6\left(0.2e^{-0.2t}\right) - 5000$$
$$= 9.8 \times 10^5 e^{-0.2t} - 5000$$

$P'(t)$ is never undefined. Let $P'(t) = 0$.

$$9.8 \times 10^5 e^{-0.2t} - 5000 = 0$$
$$e^{-0.2t} = \frac{5000}{9.8 \times 10^5}$$
$$e^{-0.2t} \doteq 0.005102041$$
$$-0.2t = \ln(0.005102041)$$
$$t \doteq 26$$

To determine the maximum profit, we evaluate

$$P(26) = 4.9 \times 10^6\left(1 - e^{-0.2(26)}\right) - 30\,000 - 5000(26)$$
$$\doteq 4\,713\,000$$
$$P(0) = 4.9 \times 10^6\left(1 - e^0\right) - 30\,000 - 0$$
$$= -30\,000 \text{ (they're losing money!)}$$
$$P(30) = 4.9 \times 10^6\left(1 - e^{-0.2(30)}\right) - 30\,000 - 5000(30)$$
$$\doteq 4\,708\,000.$$

The maximum profit of \$4 713 000 occurs when the ad campaign runs for 26 days.

Exercise 8.4

Part A

technology

1. Use your calculator to graph each of the following functions. From the graph, find the absolute maximum and absolute minimum values of the given functions on the indicated intervals.
   a. $f(x) = e^{-x} - e^{-3x}$ on $0 \leq x \leq 10$
   b. $g(t) = \dfrac{e^t}{1 + \ln t}$ on $1 \leq t \leq 12$
   c. $m(x) = (x + 2)e^{-2x}$ on $-4 \leq x \leq 4$
   d. $s(t) = \ln\left(\dfrac{t^2 + 1}{t^2 - 1}\right) + 6\ln t$ on $1.1 \leq t \leq 10$

2. a. Use the maximum and minimum algorithm to determine the absolute maximum and minimum values for the functions in Question 1.
   b. Explain which approach is easier to use for the functions that were given in Question 1.

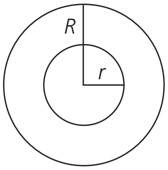
3. A small self-contained forest was studied for squirrel population by a biologist. It was found that the forest population, P, was a function of time, t, where t was measured in weeks. The function was $P = \frac{20}{1 + 3e^{-0.02t}}$.
   a. Find the population at the start of the study when $t = 0$.
   b. The largest population the forest can sustain is represented mathematically by the limit as $t \to \infty$. Determine this limit.
   c. Determine the point of inflection.
   d. Graph the function.
   e. Explain the meaning of the point of inflection in terms of squirrel population growth.

Part B

4. The net monthly profit from the sale of a certain product is given (in dollars) by the formula $P(x) = 10^6\left[1 + (x - 1)e^{-0.001x}\right]$, where x is the number of items sold.
   a. Find the number of items that yield the maximum profit. At full capacity, the factory can produce 2000 items per month.
   b. Repeat part **a**, assuming that, at most, 500 items can be produced per month.

5. Suppose the revenue (in thousands of dollars) for sales of x hundred units of an electronic item is given by the function $R(x) = 40x^2e^{-0.4x} + 30$, where the maximum capacity of the plant is eight hundred units. Determine the number of units to produce so the revenue is a maximum.

6. In a telegraph cable, the speed of the signal is proportional to $v(x) = x^2 \ln\left(\frac{1}{x}\right)$, where x is the ratio of the radius r of the cable's core to the overall radius R. Find the value of x that maximizes the speed of the signal. For technical reasons, it is required that $\frac{R}{10} \le r \le \frac{9R}{10}$.



7. A student's intensity of concentration, on a scale of 0 to 1, is given by $C(h) = 1 + h(\ln h)^2$, where h is the time of the study session in hours, and where $0 \le h \le 1$. For a student involved in a 45-minute study session, at what time is the intensity of study at its lowest level?

8. A rumour spreads through a population in such a way that t hours after its beginning, the percentage of people involved in passing on the rumour is given by $P(t) = 100(e^{-1} - e^{-4t})$. What is the highest percentage of people involved in spreading the rumour within the first 3 h? When does this occur?

9. Small countries trying to rapidly develop an industrial economy often try to achieve their objectives by importing foreign capital and technology. Statistics Canada data shows that when Canada attempted this strategy from 1867 to 1967, the amount of U.S. investment in Canada increased from about $\$15 \times 10^6$ to $\$280\,305 \times 10^6$. This increase in foreign investment can be represented by the simple mathematical model $C = 0.015 \times 10^9 e^{0.07533t}$, where t represents the number of years starting with 1867 as zero and C represents the total capital investment from U.S. sources.

   a. Graph the curve for the 100-year period.

   b. Compare the growth rate of U.S. investment in 1947 to 1967.

   c. Find the growth rate of investment in 1967 as a percentage of the amount invested.

   technology

   d. If this model is used up to 1977, calculate the total U.S. investment and the growth rate.

   e. Use the Internet to determine the actual amount of total U.S. investment in 1977, and calculate the error in the model.

   f. If the model is used up to 2007, calculate the expected U.S. investment and the expected growth rate.

10. If a drug is injected into the body, the concentration C in the blood at time t is given by the function $C(t) = \frac{k}{b-a}\left(e^{-at} - e^{-bt}\right)$, where a, $b(b > a)$, and k are positive constants that depend on the drug. At what time does the largest concentration occur?

11. A colony of bacteria in a culture grows at a rate given by $N = 2^{\frac{t}{5}}$, where N is the number of bacteria t minutes from time of starting. The colony is allowed to grow for 60 min, at which time a drug is introduced that kills the bacteria. The number of bacteria killed is given by $K = e^{\frac{t}{3}}$, where K bacteria are killed at time t minutes.

   a. Determine the maximum number of bacteria present and the time at which this occurs.

   b. Determine the time at which the bacteria colony is obliterated.

12. A student is studying for two different exams. Because of the nature of the courses, the measure of study effectiveness for the first course is $0.5\left(10 + te^{-\frac{t}{10}}\right)$ while the measure for the second course is $0.6\left(9 + te^{-\frac{t}{20}}\right)$. The student is prepared to spend 30 h in total in preparing for the exams. If E_1 is the first measure and E_2 is the second, then $f(t) = E_1 + E_2$, where $E_1 = 0.6\left(9 + te^{-\frac{t}{20}}\right)$ and $E_2 = 0.5\left(10 + te^{-\frac{t}{10}}\right)$. How should this time be allocated so as to maximize total effectiveness?

Part C

13. Suppose that in Question 12 the student has only 25 h to study for the two exams. Is it possible to determine the time to be allocated to each exam? If so, how?

14. Although it is true that many animal populations grow exponentially for a period of time, it must be remembered that eventually the food available to sustain the population will run out and at that point the population will decrease through starvation. Over a period of time, the population will level out to the maximum attainable value, L. One mathematical model that will describe a population that grows exponentially at the beginning and then levels off to a limiting value L is the **logistic model**. The equation for this model is $P = \frac{aL}{a + (L - a)e^{-kLt}}$, where the independent variable t represents the time and P represents the size of the population. The constant a is the size of the population at time $t = 0$, L is the limiting value of the population, and k is a mathematical constant.

    a. Suppose a biologist starts a cell colony with 100 cells and finds that the limiting size of the colony is 10 000 cells. If the constant $k = 0.0001$, draw a graph to illustrate this population, where t is in days.

    b. At what point in time does the cell colony stop growing exponentially? How large is the colony at this point?

    c. Compare the growth rate of the colony at the end of day 3 to the end of day 8. Explain what is happening.

Section 8.5 — Logarithmic Differentiation

The derivatives of most functions involving exponential and logarithmic expressions can be determined with the techniques we have developed. However, a function such as $y = x^x$ poses new problems. The Power Rule cannot be used because the exponent is not a constant.

The method of determining the derivative of an exponential function also can't be used because the base isn't a constant. What can be done?

It is frequently possible in functions presenting special difficulties to simplify the situation by employing the properties of logarithms. We say that we are using *logarithmic differentiation*.

EXAMPLE 1 Determine $\frac{dy}{dx}$ for the function $y = x^x$, $x > 0$.

Solution

Take natural logarithms of each side: $\ln y = x \ln x$.

Differentiate each side: $\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln x = 1 + \ln x$.

Then $\frac{dy}{dx} = y(1 + \ln x)$

$= x^x(1 + \ln x)$.

This technique of logarithmic differentiation also works well to help simplify a function with many factors and powers before the differentiation takes place.

We can use logarithmic differentiation to prove the Power Rule, $\frac{d}{dx}(x^n) = nx^{n-1}$, for all real values of n. (In previous chapters, we proved this rule for positive integer values of n and we have been cheating a bit in using the rule for other values of n.)

Given the function $y = x^n$, for any real values of n, determine $\frac{dy}{dx}$. To solve this, we take the natural logarithm of both sides of this expression and get

$$\begin{aligned}\ln y &= \ln x^n\\ &= n \ln x.\end{aligned}$$

Differentiating both sides with respect to x, using implicit differentiation and remembering that n is a constant, we get

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= n\frac{1}{x}\\ \frac{dy}{dx} &= ny\frac{1}{x}\\ &= nx^n\frac{1}{x}\\ &= nx^{n-1}.\end{aligned}$$

Therefore, $\frac{d}{dx}(x^n) = nx^{n-1}$ for any real value of n.

EXAMPLE 2 For $y = (x^2 + 3)^x$, find $\frac{dy}{dx}$.

Solution

Here we can use logarithmic differentiation again. Take the natural logarithm of both sides of the equation:

$$\begin{aligned} y &= (x^2 + 3)^x \\ \ln y &= \ln(x^2 + 3)^x \\ &= x\ln(x^2 + 3). \end{aligned}$$

Differentiate both sides of the equation with respect to x, using implicit differentiation on the left side and the Product and Chain Rules on the right side.

Therefore, $\frac{1}{y}\frac{dy}{dx} = 1 \bullet \ln(x^2 + 3) + x\left(\frac{1}{x^2 + 3}(2x)\right)$

$$\begin{aligned} \frac{dy}{dx} &= y\left[\ln(x^2 + 3) + x\left(\frac{2x}{x^2 + 3}\right)\right] \\ &= (x^2 + 3)^x\left[\ln(x^2 + 3) + x\left(\frac{2x}{x^2 + 3}\right)\right]. \end{aligned}$$

Logarithmic differentiation is useful when the function that we wish to differentiate contains a power with variables in both the base and the exponent.

You will recognize logarithmic differentiation as the method used in the previous section, and its use makes memorization of many formulas unnecessary. It also allows for complicated functions to be handled much more easily.

EXAMPLE 3 Given $y = \frac{(x^4 + 1)\sqrt{x + 2}}{}$, determine $\frac{dy}{dx}$ at $x = -1$.

Solution

While it is possible to find $\frac{dy}{dx}$ using a combination of the Product, Quotient, and Chain Rules, this process is awkward and time-consuming. Instead, before differentiating, we take the natural logarithm of both sides of the equation.

Since $y = \frac{(x^4 + 1)\sqrt{x + 2}}{}$,

$$\begin{aligned} \ln y &= \ln\left[\frac{(x^4 + 1)\sqrt{x + 2}}{}\right] \\ &= \ln(x^4 + 1) + \ln\sqrt{x + 2} - \ln(2x^2 + 2x + 1) \\ &= \ln(x^4 + 1) + \frac{1}{2}\ln(x + 2) - \ln(2x^2 + 2x + 1). \end{aligned}$$

The right side of this equation looks much simpler. We can now differentiate both sides with respect to x, using implicit differentiation on the left side.

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \frac{1}{x^4 + 1}(4x^3) + \frac{1}{2}\,\frac{1}{x + 2} - \frac{1}{2x^2 + 2x + 1}(4x + 2) \\ \frac{dy}{dx} &= y\left[\frac{4x^3}{x^4 + 1} + \frac{1}{2(x + 2)} - \frac{4x + 2}{2x^2 + 2x + 1}\right] \\ &= \frac{(x^4 + 1)\sqrt{x + 2}}{}\left[\frac{4x^3}{x^4 + 1} + \frac{1}{2(x + 2)} - \frac{4x + 2}{2x^2 + 2x + 1}\right] \end{aligned}$$

While this derivative is a very complicated function, the process of finding the derivative is straightforward, using only the derivative of the natural logarithm function and the Chain Rule.

We do not need to simplify this in order to determine the value of the derivative at $x = -1$.

For $x = -1$, $\frac{dy}{dx} = \frac{(1+1)\sqrt{1}}{}\left[\frac{-4}{1+1} + \frac{1}{2(-1+2)} - \frac{-4+2}{2-2+1}\right]$

$$= 2\left[-2 + \frac{1}{2} + 2\right]$$
$$= 1.$$

Exercise 8.5

Part A

1. Differentiate each of the following.

   a. $y = x^{\sqrt{10}} - 3$ b. $f(x) = 5x^{3\sqrt{2}}$ c. $s = t^{\pi}$ d. $f(x) = x^e + e^x$

Knowledge/Understanding

2. Use the technique of Logarithmic Differentiation to find the derivative for each of the following:

   a. $y = x^{\ln x}$ b. $y = \frac{(x+1)(x-3)^2}{(x+2)^3}$

   c. $y = x^{\sqrt{x}}$ d. $s = \left(\frac{1}{t}\right)^t$

3. a. If $y = f(x) = x^x$, evaluate $f'(e)$.

   b. If $s = e^t + t^e$, find $\frac{ds}{dt}$ when $t = 2$.

   c. If $f(x) = \frac{(x-3)^2\sqrt[3]{x+1}}{}$, determine $f'(7)$.

4. Find the equation of the tangent to the curve defined by $y = x^{(x^2)}$ at the point where $x = 2$.

Part B

5. If $y = \frac{1}{}$, find the slope of the tangent to the curve at the point where $x = 0$.

Application

6. Determine the points on the curve defined by $y = x^{\frac{1}{x}}$, $x > 0$, where the slope of the tangent is zero.

7. If tangents to the curve defined by $y = x^2 + 4 \ln x$ are parallel to the line defined by $y - 6x + 3 = 0$, find the points where the tangents touch the curve.

Thinking/Inquiry/ Problem Solving

8. The tangent at point A (4, 16) to the curve defined by $y = x^{\sqrt{x}}$ is extended to cut the x-axis at B and the y-axis at C. Determine the area of ΔOBC where O is the origin.

Part C

9. The position of a particle that moves on a straight line is given by $s(t) = t^{\frac{1}{t}}$ for $t > 0$.

   a. Find the velocity and acceleration.

   b. At what time, t, is the velocity zero? What is the acceleration at that time?

Thinking/Inquiry/ Problem Solving

10. Without using a calculator, determine which number is larger, e^{π} or π^{e}. (*Hint:* Question 9 can help.) Verify your work with a calculator.

Key Concepts Review

In this chapter, we introduced a new base for exponential and logarithmic functions, namely e, where $e \approx 2.718281$. Approximations to the value of e can be calculated using one of the following two limits:

- $e = \lim_{h \to 0} (1 + h)^{\frac{1}{h}}$
- $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

New Derivative Rules for Exponential and Logarithmic Functions

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}\ln x = \frac{1}{x}, x > 0$
- $\frac{d}{dx}(b^x) = b^x \ln b$
- $\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)}(\ln b)g'(x)$
- $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$
- $\frac{d}{dx}(\log_b g(x)) = \frac{g'(x)}{g(x) \ln b}$

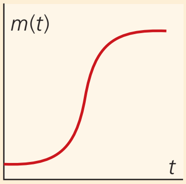
The only rules that need to be memorized are those in the shaded area. The other rules can be determined from the first two, using properties of exponential and logarithmic functions along with logarithmic differentiation.

Career Link wrap-up investigate and apply

CHAPTER 8: RATE-OF-CHANGE MODELS IN MICROBIOLOGY

To combat the widespread problem of soil and groundwater contamination, scientists and engineers have investigated and engineered bacteria capable of destroying environmental toxicants. The use of bacteria in environmental clean-ups, known as bioremediation, has been proven effective in destroying toxic compounds ranging from PCBs to gasoline additives such as benzene. An environmental engineer conducting a lab study found the growth in mass of a quantity of bioremediation bacteria follows a "logistic" growth pattern. The logistic model is characterized by the familiar "S"-shaped graph and equation as follows:

$$m_b(t) = \frac{L}{1 + \left(\frac{L - m_0}{m_0}\right)e^{-Lkt}}$$



where $m_b(t)$ is the mass of bacteria at time t, L is bounded/maximum mass, k is the growth constant, and m_0 is the initial mass. The model can be constructed by substituting values of m_0, L, and a known ordered pair for (t, m_b) into the equation and solving for k.

The engineer conducting the study found that starting from an initial mass of 0.2 kg, the bacteria grow to a maximum mass of 2.6 kg following a logistic growth pattern. The mass after five days for this experiment was 1.5 kg. The engineer has modelled the mass of contaminant remaining in kilograms as

$$m_c(t) = -\log_3(\sqrt{t} + 1) + 2.5$$

where $m_c(t)$ is the mass of contaminant remaining (kilograms) in t days.

a. Develop the logistic growth function model for the bacterial mass.

b. Like humans, many bacteria also need oxygen to survive. The oxygen demand for bacteria is

$$D_{O_2} = 10(m_c)\left(\frac{dm_b}{dt}\right) \text{ [litres per hour]}$$

What is the oxygen demand after five days?

c. The experiment is re-inoculated (new bacteria added) when the amount of contamination has reached 50% of the initial mass. When must the new bacteria be added, and how quickly is the contamination being destroyed at this time?

Review Exercise

1. Use the Chain Rule and the appropriate derivative rule for a logarithmic or exponential function to find the derivative for each of the following:

   a. $y = e^{2x+3}$ b. $s = \ln(t^3 + 1)$ c. $f(x) = \ln(x^3 - 3x^2 + 6x)$

   d. $y = e^{-3x^2+5x}$ e. $y = \ln(e^x + e^{-x})$ f. $y = 2^{e^x}$

2. Use the appropriate derivative rules to find the derivative for each of the following. If necessary, use logarithmic differentiation.

   a. $y = xe^x$ b. $y = \frac{x\ln x}{e^x}$

   c. $s = \sqrt{t^4 + 2} \bullet \ln(3t)$ d. $y = \frac{(x + 2)(x - 4)^5}{(2x^3 - 1)^2}$

   e. $s = \frac{e^t - 1}{e^t + 1}$ f. $y = \left(\sqrt{x^2 + 3}\right)^{e^x}$

   g. $y = \left(\frac{30}{x}\right)^{2x}$ h. $e^{xy} = \ln(x + y)$

3. For each of the following functions, solve the equation $f'(x) = 0$.

   a. $f(x) = \frac{e^x}{x}$ b. $f(x) = \left[\ln(3x^2 - 6x)\right]^4$

4. a. If $g(t) = 10^t \log_{10} t$, then evaluate $g'(10)$.

   b. If $f(x) = xe^{-2x}$, then find $f'\left(\frac{1}{2}\right)$.

5. Find the second derivative for each of the following:

   a. $s = t\ln t$ b. $y = xe^{10x}$

6. Find the equation of the tangent to the curve defined by $y = \frac{\ln x^2}{x}$ at the point where $x = 4$.

7. If $y = \frac{e^{2x} - 1}{e^{2x} + 1}$, prove that $\frac{dy}{dx} = 1 - y^2$.

8. Find the value of k in the equation $y = e^{kx}$ so that y is a solution of

   a. $y' - 7y = 0$. b. $y'' - 16y = 0$.

   c. $y''' - y'' - 12y' = 0$.

9. Find the equation of the tangent to the curve defined by $y = x - e^{-x}$ that is parallel to the line represented by $3x - y - 9 = 0$.

10. Find an equation for the normal line to the curve defined by $y = \frac{e^x}{1 + \ln x}$ at the point where $x = 1$.

11. The number, N, of bacteria in a culture at time t is given by $N = 2000\left[30 + te^{-\frac{t}{20}}\right]$.

    a. When is the rate of change of the number of bacteria equal to zero?

    b. If the bacterial culture is placed into a colony of mice, the number of mice, M, that become infected is related to the number of bacteria present by the equation $M = \sqrt[3]{N + 1000}$. After ten days, how many mice are becoming infected per day?

12. The measure of effectiveness of a medicine on a scale of 0 to 1 is given by $g(t) = \frac{\ln(t^3)}{2t}$, where t is the time in hours after administering the medicine and $t > 1$. Determine the maximum measure of effectiveness of this medicine and the time at which it is reached.

13. Some psychologists model a child's ability to memorize by a function of the form $m(t) = t\ln(t) + 1$ for $0 < t \leq 4$, where t is time, measured in years. Determine when a child's ability to memorize is highest and when it is lowest.

14. The concentrations of two medicines in the blood stream t hours after injection are $c_1(t) = te^{-t}$ and $c_2(t) = t^2e^{-t}$.

    a. Which medicine has the larger maximum concentration?

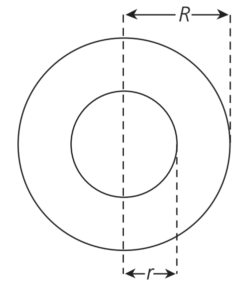
    b. Within the first half-hour, which medicine has the larger maximum concentration?

15. One model of a computer disk storage system uses the function $T(x) = N\left(k + \frac{c}{x}\right)p^{-x}$ for the average time needed to send a file correctly by modem (including all re-transmission of messages in which errors were detected), where x is the number of information bits, p is the (fixed) probability that any particular file will be received correctly, and N, k, and c are positive constants. Use the values $p = 0.9$, $N = 10$, and $c = 1$ to answer the following questions:

    a. Find $T'(x)$.

    b. For what value of x is $T(x)$ minimized?

16. In a telegraph cable, the speed of the signal is given by $v(x) = kx^2 \ln\left(\frac{1}{x}\right)$, where k is a positive constant and x is the ratio of the radius r of the cable's core to the overall radius R.



   a. For $k = 2$, determine the speed of the signal if $x = \frac{1}{2}$.

   b. For $k = 2$, at what rate is the speed changing when $x = \frac{1}{2}$?

17. The function $C(t) = K\left(e^{-2t} - e^{-5t}\right)$, where K is a positive constant, can be used to model the concentration at time t, in days, of a drug injected into the bloodstream.

   a. Evaluate $\lim_{t \to \infty} C(t)$.

   b. Find $C'(t)$, the rate at which the drug is cleared from circulation.

   c. When is this rate equal to zero?

Chapter 8 Test

Achievement Category	Questions
Knowledge/Understanding	All questions
Thinking/Inquiry/Problem Solving	8, 9
Communication	8
Application	3, 5, 6, 7, 8

1. Find the derivative $\frac{dy}{dx}$ for each of the following:
   a. $y = e^{-2x^2}$ b. $y = \ln(x^2 - 6)$ c. $y = 3^{x^2+3x}$
   d. $y = \frac{e^{3x} + e^{-3x}}{2}$ e. $y = (4x^3 - x)\log_{10}(2x - 1)$ f. $y = \frac{\ln(x + 4)}{x^3}$

2. If $f(t) = \ln(3t^2 + t)$, determine $f'(2)$.

3. If $y = x^{\ln x}$, $x > 0$, find the slope of the tangent when $x = e$.

4. Determine $\frac{dy}{dx}$ for the relation $x^2y + x\ln x = 3y$, where $x > 0$.

5. If $e^{xy} = x$, determine $\frac{dy}{dx}$ at $x = 1$.

6. For what values of A does the function $y = e^{Ax}$ satisfy the equation $y'' + 3y' + 2y = 0$?

7. Find the equation of the straight line that is normal to the curve defined by $y = 3^x + x\ln x$ at point $A(1, 3)$.

8. The velocity of a particular particle that moves in a straight line under the influence of forces is given by $v(t) = 10e^{-kt}$, where k is a positive constant and $v(t)$ is in cm/s.
   a. Show that the acceleration of this particle is proportional to (a constant multiple of) its velocity. Explain what is happening to this particle.
   b. At time $t = 0$, what is the initial velocity of the particle?

c. At what time is the velocity equal to half of the initial velocity? What is the acceleration at this time?

9. A manufacturer can produce jackets at a cost of \$50 per jacket. If he produces p jackets weekly and all the jackets are sold, the revenue is $R(p) = 4000[e^{0.01(p-100)} + 1]$. Weekly production must be at least 100 and cannot exceed 250.

   a. At what price should the manufacturer sell the jackets to maximize profit?

   b. What is the maximum weekly profit?